Paying Attention to Spatial Reasoning

K–12

Support Document for Paying Attention to Mathematics Education

Contents

Paying Attention to Spatial Reasoning

"Spatial thinking is integral to everyday life. People, natural objects, human-made objects, and human-made structures exist somewhere in space, and the interactions of people and things must be understood in terms of locations, distances, directions, shapes, and patterns."

(National Research Council, 2006, p. 5)

Paying Attention to Mathematics Education provided an overview of what it would take to help Ontario students make – and sustain – gains in their learning and understanding of mathematics. It outlined seven foundational principles for planning and implementing improvements and gave examples of what each principle would involve.

This document gets more concrete by focusing on a particular area of mathematics. Other support documents will explore other key topics in mathematics teaching and learning.

Seven Foundational Principles for Improvement in Mathematics, K–12

- Focus on mathematics.
- Coordinate and strengthen mathematics leadership.
- Build understanding of effective mathematics instruction.
- Support collaborative professional learning.
- Design a responsive mathematics learning environment.
- Provide assessment and evaluation in mathematics.
- Facilitate access to mathematics learning resources.

What Is Spatial Reasoning?

"Spatial thinking is powerful. It solves problems by managing, transforming, and analyzing data, especially complex and large data sets, and by communicating the results of those processes to one's self and to others."

(National Research Council, 2006, p. 5)

Spatial thinking, or reasoning, involves the location and movement of objects and ourselves, either mentally or physically, in space. It is not a single ability or process but actually refers to a considerable number of concepts, tools and processes (National Research Council, 2006).

According to the National Research Council (2006), spatial thinking involves three components: "concepts of space, tools of representation, and processes of reasoning" (p. 3). It involves understanding relationships within and between spatial structures and, through a wide variety of possible representations (from drawings to computer models), involves the means to communicate about them. When a child rotates a rectangular prism to fit into the castle she is building at the block centre, she is employing spatial reasoning, as is the student who uses a diagram of a rectangle to prove that the formula for finding the area of a triangle is $\frac{1}{2}b \times h$. Spatial reasoning vitally informs our ability to investigate and solve problems, especially non-routine or novel problems, in mathematics.

The Ontario curriculum combines spatial sense and geometry into one strand (as do many curricula around the world) because spatial sense and geometry are inherently linked. Geometry, which translates roughly as "measure of the Earth," deals directly with measuring and moving objects in space. Geometry is the foundation of mathematics as we know it today; it was developed to explain phenomena and solve problems that bore directly on daily life, such as how to measure time or navigate across the sea. Spatial thinking gave birth to the earliest forms of sophisticated mathematical thinking. And yet, in spite of its importance, research has shown that, in North America, geometry receives the least amount of time compared with other strands in classroom instruction (see Bruce, Moss & Ross, 2012; Clements & Sarama, 2011).

We are just beginning to understand the interplay between spatial reasoning and mathematics learning. We know that by focusing on spatial thinking, we can tap into a diversity of student strengths. A focus on spatial thinking allows mathematics to become a more visual endeavour and connects with what "real" mathematicians do when they are exploring patterns in the world and making discoveries. By exploring the spatial aspects of mathematics, we make it more accessible, more engaging and more relevant. Albert Einstein conceived his theory of relativity, which produced possibly the most familiar equation of all time ($E = mc^2$), by visualizing himself riding a beam of light. Stephen Hawking has explained that "by losing the finer dexterity of my hands, I was forced to travel the universe in my mind, and try to visualize the ways in which it worked" (Johnson, 2014). We need to continually foster students' creative engagement in mathematics, and paying attention to spatial thinking is a key to doing so.

"First, remember that spatial intelligence has evolutionary and adaptive importance. Any mobile organism must be able to navigate in its world to survive and must represent the spatial environment in order to do so."

(Newcombe & Frick, 2010, p. 102)

Spatial reasoning can involve

Why Is Spatial Reasoning Important?

"Research on spatial reasoning substantiates the critical importance of spatial reasoning abilities in geometry, measurement and problem solving both early in students' mathematics experiences as well as later in high school and beyond, especially in STEM areas."

(Shumway, 2013, p. 50)

Who needs to think spatially? Besides the fact that we all need to navigate our way around in a three-dimensional physical world, careers in the sciences, technology, engineering and mathematics (STEM) require strong spatial skills. In fact, research has shown that spatial ability is a predictor of success in these areas (see Newcombe, 2010, 2013; Wai, Lubinski & Benbow, 2009). Spatial thinking is also strongly employed in many of the arts. (Recently, there has been a movement to add the arts to the STEM category, creating the acronym STEAM.) Architecture, graphic design, computer sciences, biology, physics, chemistry, geology, geography and even medicine (consider the spatial reasoning required to understand various ways of mapping the body, such as x-rays and MRIs) all require strong spatial skills.

Perhaps owing to its complexity and because we still have much to understand about it, instructional supports for the explicit teaching and learning of spatial strategies are currently lacking. The good news is that this is changing. In its report *Learning to Think Spatially*, the National Research Council (2006) issued a call for action in education: that we recognize spatial thinking as important not only across mathematical strands but also across subject areas, and that educational researchers and system leaders develop better understandings and supports to foster spatial literacy in students. The National Research Council describes the current situation as a "major blind spot" in education and maintains that, without explicit attention to spatial thinking, the concepts, tools and processes that underpin it "will remain locked in a curious educational twilight zone: extensively relied on across the K–12 curriculum but not explicitly and systematically instructed in any part of the curriculum" (p. 7).

Turning to Research: Reasons to Pay Attention to Spatial Reasoning in Mathematics

"Most of us have been taught to think and talk about the world using words, lists, and statistics. These are useful tools but they do not come close to telling the full story. Thinking spatially opens the eye and mind to new connections, new questions, and new answers."

(Center for Spatial Studies, UCSB, n.d.)

Spatial thinking plays a fundamental role throughout the K–12 curriculum. Whether it is the learning of science, mathematics, art, physical education or literacy, spatial thinking skills are important. For example, high school chemistry requires students to understand the spatial structure of molecules. Physical activity calls on students' awareness of their body's position in space and with respect to other objects. Art – of all forms – is filled with opportunities to engage our spatial skills, whether it is playfully manipulating shapes and forms while painting or representing musical notes spatially. Of particular importance, however, is the role of spatial thinking in mathematics education. Research findings across education, psychology and neuroscience reveal a close link between spatial thinking and mathematics learning and achievement.

1. Spatial thinking is critical to mathematical thinking and achievement.

"The relation between spatial ability and mathematics is so well established that it no longer makes sense to ask whether they are related."

(Mix & Cheng, 2012, p. 206)

Nearly a century of research confirms the close connection between spatial thinking and mathematics performance (Mix & Cheng, 2012). In general, people with strong spatial skills also tend to perform well in mathematics. Moreover, the strength of this connection does not appear to be limited to any one strand of mathematics. Researchers have found evidence to suggest that spatial thinking plays an important role in arithmetic, word problems, measurement, geometry, algebra and calculus. Exactly how spatial ability connects with mathematical ability – and exactly what kinds of spatial ability connect with which mathematical skills – are areas for future research. Researchers are particularly interested in how spatial ability supports numeric proficiency, and recent research in mathematics education, psychology and even neuroscience is attempting to map these relationships. It also appears, for example, that spatial ability is connected to understanding numeric quantities and to early numeracy performance (for a summary of this research, see Drefs and D'Amour, 2014). Research also shows that spatial skills might be predictive of later mathematics achievement. For example, a recent longitudinal study with three-year-olds found evidence that spatial skills were even more important than early mathematics skills and vocabulary at predicting mathematics performance at the age of five (Farmer et al., 2013). Studies with adolescents further highlight the role of spatial thinking in predicting later academic success. In a longitudinal study involving 400,000 students, Wai and colleagues (2009) found that spatial skills assessed in high school predicted which students would later enter and succeed in disciplines related to science, technology, engineering and mathematics. Moreover, spatial thinking was a better predictor of mathematics success than either verbal or mathematical skills. Taken together, the above research findings paint a clear picture: when it comes to mathematics, spatial thinking matters.

2. Spatial thinking is malleable and can be improved through education and experience.

There is a widespread belief that spatial thinking skills are fixed – either you are a spatial thinker or you are not. This is a misconception. Spatial thinking is made up of many skills, and for this reason, it is possible to excel in certain aspects of spatial thinking, such as navigational skills, while demonstrating relative weaknesses in other areas, such as visualization skills. Even more important, however, is the finding that spatial thinking skills can be improved with practice. A recent meta-analysis – summarizing more than two decades of research on spatial training – revealed that spatial thinking can be improved through an assortment of activities and across all age groups (Uttal et al., 2013). Specific approaches shown to improve spatial thinking skills include puzzle play, video games (e.g., Tetris), block building, practising spatial activities, art and design tasks, and in-class lessons and activities designed to support and develop students' spatial thinking skills. Many studies have shown that improvements in one type of spatial reasoning task often transfer to other types of tasks (even novel, unfamiliar tasks); figuring out what these connections are (exactly how, what and why these improvements occur) is an exciting area for future research. More recent research efforts are now underway to determine whether improvements in spatial thinking lead to improvements in mathematics performance. In the first study to suggest that this might be the case, Cheng and Mix (2012) had six- to eight-year-olds complete mathematics and spatial tests and then engage in 45 minutes of either mental rotation practice (spatial thinking group) or crossword puzzles (control group). Children were then re-tested on the same spatial and mathematics tasks. Compared with those in the control group, children in the spatial thinking group demonstrated significant improvements in their calculation skills, especially on missing-term problems (e.g., 5 + ____ = 8). While more studies are needed to firmly establish a causal relationship between spatial thinking and mathematics performance, the findings from this study, and others currently underway, provide reasons to be optimistic about the widespread benefits of learning to think spatially. In sum, sufficient evidence suggests that spatial thinking skills can be improved with practice. Given the close link between spatial and mathematical skills, "we can expect that spatial instruction will have a 'two-for-one' effect, yielding benefits in mathematics as well" (Verdine et al., 2013, p. 13). Perhaps just as effective, if not more, is an approach that integrates and supports students' spatial skills throughout mathematics instruction.

3. Schools play an important role in fostering spatial reasoning.

Studies have shown that children's spatial abilities grow over the school year but stall in the summer months (Huttenlocher, Levine & Vevea, 1998). This result shows that not only can spatial thinking be improved but also that something that we are already doing in schools is improving it (Newcombe, 2010). Research is emerging to show what that "something" is and what classroom strategies we can use to enhance children's spatial reasoning further. Finding these strategies is especially important for underserved populations and for addressing issues of equity through education (see below).

Socioeconomic issues

"As a group, children from disadvantaged, low-income families perform substantially worse in mathematics than their counterparts from higher-income families. Minority children are disproportionately represented in low-income populations, resulting in significant racial and social-class disparities in mathematics learning linked to diminished learning opportunities. The consequences of poor mathematics achievement are serious for daily functioning and for career advancement."

(Jordan & Levine, 2009, p. 60)

Improving access to mathematics is a moral imperative for educators, and it has implications for social equity, especially for children from neighbourhoods with low socioeconomic status (SES). The connection between socioeconomic status and mathematics success in school is well established; students from low-SES backgrounds are less likely to do well in mathematics or to go on to higher mathematics than their higher-SES peers (Jordan & Levine, 2009). Lack of success in mathematics is a barrier that narrows career options in later life and prevents many students from attaining careers that would help them to break the cycle of poverty (we know that many higher-paying careers require at least some mathematics background). Of course, beyond improving future employment opportunities, our work as educators is to enable all students to realize their full potential. Attention to spatial reasoning can provide additional entry points into mathematics for all children and improve their prospects for the future and success in later life.

Gender inequities

Females are still greatly underrepresented in STEM careers, and males outperform females on spatial reasoning tests. However, Nora Newcombe (2010) points out that "average sex differences do not tell us about individual performance – some girls have strong spatial skills, and some boys are lacking these skills" (p. 33). The most important point for those concerned about education is that girls – and boys – can improve their spatial thinking. Research by Casey, Erkut, Ceder and Young (2008) has shown that spatial thinking opportunities, such as puzzle and block play, is "especially helpful for girls and children from schools in lower SES neighbourhoods" (in Tepylo, Moss & Hawes, 2014). The imperative is to provide high-quality opportunities for girls and boys to explore spatial ideas, processes and tools to increase their facility in spatial reasoning and increase their access to mathematics in all its forms.

Mathematics learning difficulties

Unfortunately for mathematics educators and for children, there are few appropriate resources to help students with special needs in mathematics and, indeed, no agreed-on assessments to identify special learning difficulties or disabilities (Sarama & Clements, 2009). By contrast, we know far more about literacy and difficulties in language development and processing (Ansari, 2013). One identified mathematics learning disability related to spatial cognition is called *spatial acalculia*, which is characterized by difficulties aligning numerals and reading operational signs (Mix & Cheng, 2012). But much more work needs to be done to uncover the connections between spatial ability and mathematics performance and understanding, and to give educators tools for identification and intervention. One example from the research shows how spatial deficits affect children's ability to comprehend numeric quantities and magnitudes (Sarama & Clements, 2009). This problem likely starts very early, when children start to *subitize* (from the Latin for the sudden apprehension of small quantities – think of dice and the almost-instant awareness of the quantity in particular dot arrangements without needing to count the dots). In children with mathematics learning difficulties, the ability to subitize may be delayed, and these children may need to continue to count individual items in small collections; children with this difficulty in dealing with small quantities are at serious risk in their mathematical development (Sarama & Clements, 2009).

Research on mathematics learning difficulties has identified three sub-groups: those with equal difficulties in both language and mathematics, those who lag in both subjects but are relatively stronger in mathematics, and those with learning difficulties in mathematics only. (Rourke, 1993, performed this research with 9- to 14-year-olds, but subsequent research has confirmed similar groupings in children, adolescents and adults; see Mix & Cheng, 2012.) Those with difficulties in both subjects, and those with relative strengths in math, had stronger visual-spatial abilities and weaker verbal skills than the children with math-only learning difficulties (and they consequently struggled with tasks that were highly verbal, like word problems). The group with math-only learning difficulties had much higher verbal ability but much lower spatial ability (and a host of difficulties related to mathematics performance, including reading symbols, arranging and writing numerals, following procedures, memorizing number facts and gauging the reasonableness of their responses). This research also found that the gap in spatial ability widens over time, so that "children with visuospatial deficits lose ground steadily as they age" (Mix & Cheng, 2012, p. 219).

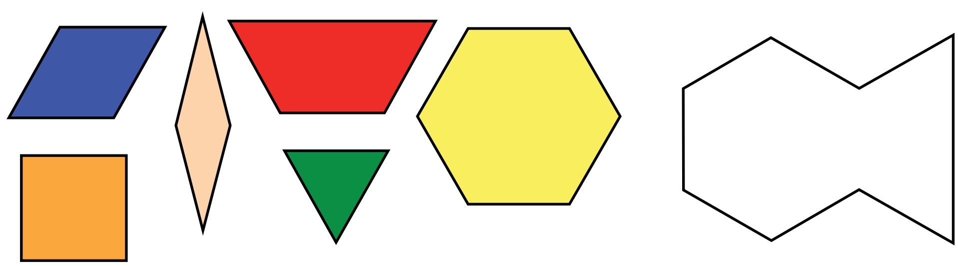
Generally speaking, although we know that strength in spatial ability is related to success in mathematics, and that weakness in spatial domains can negatively affect mathematics performance, there is much work to be done in educational and psychological research to pinpoint the exact connections, especially in ways that will help us with instruction and intervention. We do know this: early identification of children's spatial difficulties is crucial, and emerging research that connects psychology, neuroscience and mathematics education holds great promise in developing early interventions that can help level the playing field for these children.

Key Concepts in Spatial Reasoning: Exploring the Role of Spatial Visualization

Not all types of spatial thinking skills are equally related to performance in mathematics. The learning of mathematics depends on certain spatial thinking skills more than others. Spatial visualization has proven to be particularly important for mathematics learning and achievement. Spanning its influence across grades and strands, spatial visualization is a key concept that helps learners both understand and create mathematics.

What Is Spatial Visualization? Spatial visualization is a specific type of spatial thinking that involves using our imagination to "generate, retain, retrieve, and transform well-structured visual images" (Lohman, 1996, p. 98), sometimes referred to as thinking with the "mind's eye." The discovery of the structure of DNA, the theory of relativity and the invention of the motor were all described as creations borne of spatial visualization. In an attempt to foster this type of thinking in students, it is important that we have a strong grasp of what it means to engage in spatial visualization. Below are some specific examples of mathematics tasks that call on this important skill.

1. Composition and Decomposition Tasks



Using any combination of the pattern blocks above, and your visualization skills, determine the fewest number of blocks needed to fill the figure at the right.

What is the greatest number of blocks needed to fill the figure?

Composing and decomposing activities are full of opportunities to visualize possible solutions before actually carrying out the task with manipulatives.

2. Imagine This: Problem Solving with the "Mind's Eye"*

Imagine a large cube floating in front of you. The cube is made up of 64 smaller cubes and thus is a $4 \times 4 \times 4$ cube. Now, imagine that you are staring directly at the front of the cube so that all you can see is the front face of the cube – a 4×4 square face. You are now going to drill a hole through the four corner cubes that are facing you, all the way through to the back face.

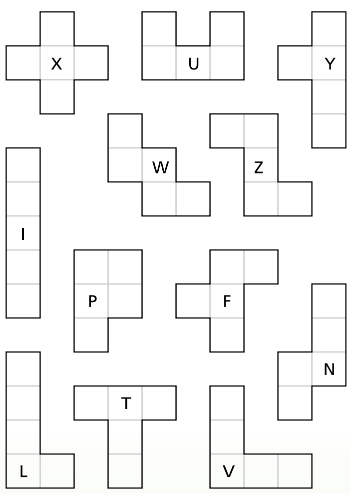
Now, imagine looking down on the cube from above – a bird's-eye view. Again, your view is such that all you see is a 4×4 square. You drill a hole through the four corner cubes all the way through to the other side.

How many of the 64 cubes do not have holes drilled through them?

Were you able to visualize the solution? To solve this problem, we must perform a number of skills that are typically involved in spatial visualization: form a mental image (a three-dimensional cube), keep the mental image in mind, change perspectives (seeing the cube from the front and top) and transform the mental image in some way (drilling holes through the corners and rotating the cube in your mind). Often, problems or solutions will call on these various aspects of spatial visualization to varying degrees. For this reason, spatial visualization can look and feel different depending on the context. At its core, however, spatial visualization requires making and retaining a mental image and changing or manipulating it in some transformative way.

Problems like the one above are often challenging at first, but with the selection of age-appropriate problems and scaffolding – along with plenty of practice – students will further develop their abilities to visualize and carry out mental problem solving.

3. Nets and Folding Exercises



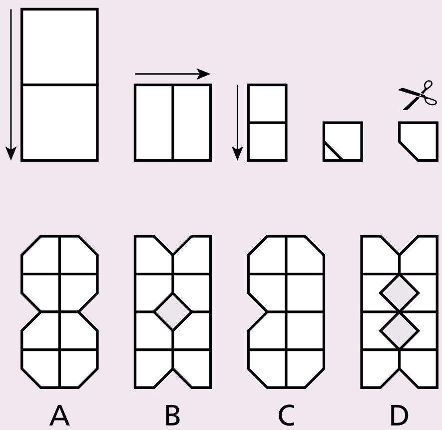
Which pentomino shapes above can be folded to form a box (i.e., open-topped cube)?

Activities that involve mentally folding a structure into a new form rely heavily on spatial visualization skills. In this activity, like many others that involve spatial visualization, it is often important to first visualize a potential solution and then to test the solution through concrete or hands-on experience (e.g., physically folding the pentomino nets to test the predictions).

* The challenge in #2 was adopted from the University of Cambridge's NRICH Enriching Mathematics Project website: http://nrich.maths.org/frontpage. Visit this website for other visualization exercises along with many other spatial activities across the grade levels.

Assessing Spatial Visualization Skills through Paper Folding

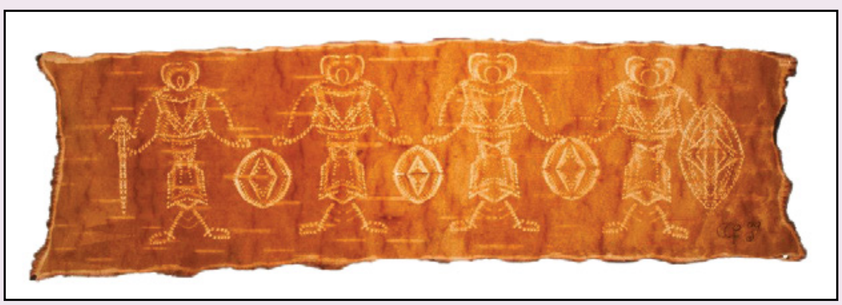
A common test of visual-spatial skills involves paper folding. To experience this type of task first-hand, try the following challenge. Note: The arrows indicate the direction of the fold, and the scissor icon represents a cut along the designated line. What piece of paper represents the finished result?



What does the finished product look like? Were you able to visualize the outcome?

True to the definition of spatial visualization, this task requires you to generate, retain and transform the folding and cutting of the piece of paper.

Interestingly, this test item is closely aligned to the skills required in the First Nation art form of birchbark biting. For millennia, First Nations peoples have created beautiful and intricate works of art through folding pieces of birchbark and biting tiny holes into the bark. The result of unfolding the bark into its original shape can be seen in the example below.



Once We Were Warriors (2009). Art by Half Moon Woman (Pat Bruderer). Reproduced with the permission of the artist.

For more information on this art form, see the video at http://www.youtube.com/watch?v=bFJaa9ndAts.

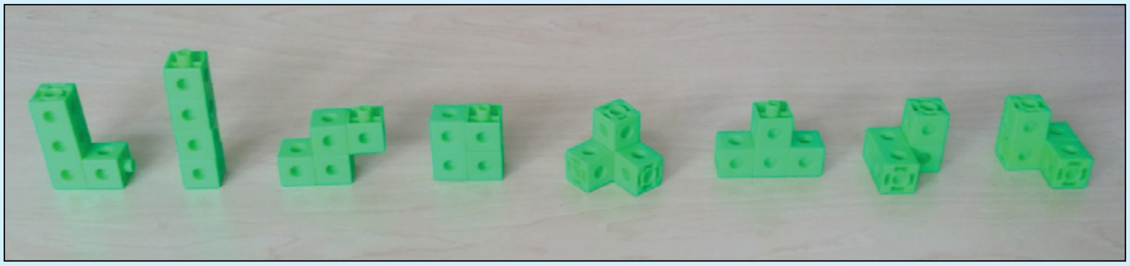
In addition to spatial visualization, mental rotation and visual-spatial working memory are two other types of spatial skills that have been shown to be highly related to mathematics learning and achievement.

Mental Rotation: Of all the spatial skills studied, mental rotation has garnered the most attention by researchers. Defined as the ability to rotate two- or three-dimensional objects in mind, mental rotation skills have been linked to performance across a variety of mathematics tasks, including arithmetic, word problems, geometry and algebra. Although mental rotation is often considered a spatial skill in itself, it is important to recognize that mental rotation is also an example of spatial visualization: another reminder that there is considerable overlap in the skills and terms that we use to define spatial thinking.

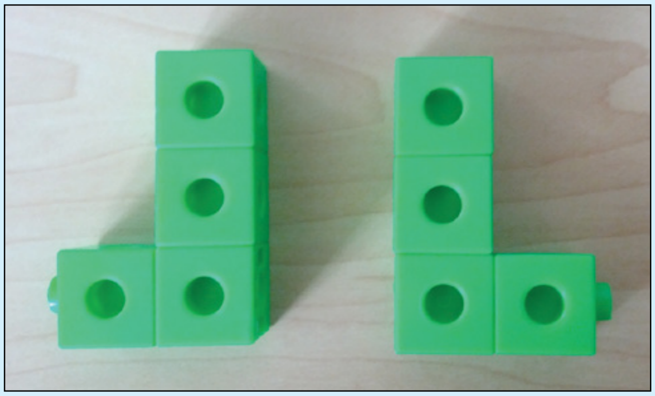
Try This!

Cube Challenge: Discovering Three-Dimensional Equivalence

Using sets of three, four or five multilink cubes, build as many unique three-dimensional figures as possible. It is recommended to first use three cubes before moving on to four and eventually five cubes (see the solution below for all possible unique figures using four cubes).



In the image above, notice that the two figures on the far right are mirror images of each other. Students (and adults!) often spend considerable time discussing and justifying why these two figures are unique three-dimensional figures. Another important concept that arises from this challenge involves that of three-dimensional equivalence (similar to the idea of two-dimensional congruence). Quite often, students think they have two unique figures, when, in fact, the two figures are equivalent but need to be reoriented (see the example below). To identify that two or more figures are equivalent often involves comparisons made possible through, first, mental rotation and, second, physical comparisons. When working with five cubes, the challenge of identifying unique figures becomes even more apparent.



Spatial Skills Involved

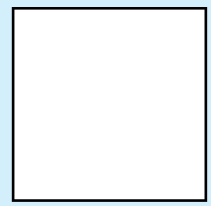
- visualization (imagining the various combinations that are possible)
- composition and decomposition of three-dimensional figures
- understanding three-dimensional equivalence through mental and physical rotations

Visual-Spatial Working Memory: Many spatial tasks call on or even depend on visual-spatial working memory. Visual-spatial working memory refers to the temporary storage (short-term memory) and manipulation of visual and spatial information. In referring back to the three spatial visualization exercises, we can see and feel our visual-spatial working memory: holding an image in mind (e.g., a triangular pattern block) and then manipulating it or transforming it in some way (e.g., visualizing how the triangle can be used – through iterations and rotations – to compose a hexagon). An emerging body of research suggests that when it comes to mathematics learning and achievement, visual-spatial working memory plays a critical role. As educators, we need to be aware that students process and remember information through both verbal and visual-spatial modalities; whether we are teaching literacy or mathematics, we need to honour both modes of thinking and learning.

Try This!

Drawing to See Shapes Differently

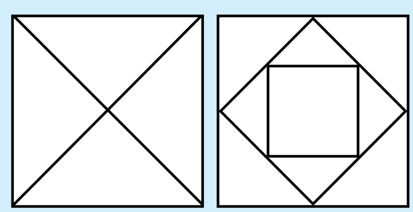
In this activity, students are required to pay attention to shape, structure and geometric relationships as they reconstruct geometric line drawings from memory (e.g., a square with a line through it). Before beginning, make sure each student has a pencil and piece of paper with a black outlined square (see image).



When the class is ready, hold up (or present on the SmartBoard) a selected geometric design (see examples below), and have the class examine it carefully for about five seconds (students should not be drawing at this point but studying the design in terms of shape, structure and geometric relationships). After five seconds have passed, hide the design and have the class try their best to reconstruct the image from memory. Hold up the design again, and have students determine whether their copy is an exact replica of the original. If required, have students make corrections to their work. End by having a class discussion, or peer-led small-group discussions, about the geometric properties of the design: How did students remember the design? How did students see the design differently (e.g., an envelope versus an X). What sort of strategies did you use to remember the design? Were there certain shapes that stood out? Are there different ways you could draw the same design? What do you notice happens when you cut a square in half along the diagonal? Through a class-wide discussion, students will come to recognize that there are many ways to see, remember and construct or deconstruct two-dimensional space.

Spatial Skills Involved

- composition or decomposition of shape and space
- proportional reasoning
- visual-spatial memory



Spatial Reasoning across Strands and Grades

"Spatial thinking is not an add-on to an already crowded school curriculum, but rather a missing link across that curriculum. Integration and infusion of spatial thinking can help to achieve existing curricular objectives."

(National Research Council, 2006, p. 7)

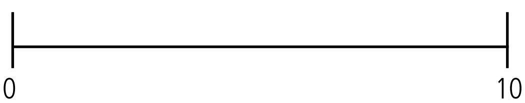
An emphasis on spatial reasoning may provide additional entry points to mathematics for many students who may have a fragile or developing concept of number, for example, but strong visual-spatial skills. And by building opportunities for students to improve their spatial thinking, we are supporting understanding in other areas of mathematics and providing experiences to foster mathematical processes, such as communication and representation.

Early intervention in mathematics is crucial, so providing young children with opportunities to develop their spatial sense is key to developing their potential as mathematics learners (Sinclair & Bruce, 2014). But we also know that spatial thinking grows in importance during adolescence as students move through an increasingly abstract curriculum into higher mathematics, as "space is more highly associated with mathematics in the higher grade levels" (Mix & Cheng, 2012, p. 219).

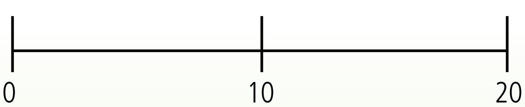
Below are some examples to get you thinking about what spatial reasoning looks like across the strands and grades.

Primary and Junior

Number Sense and Numeration: The number line can be an effective tool in fostering students' number sense. Whether students are being introduced to ideas about whole numbers or rational numbers, the number line provides a useful spatial representation of quantities and their various relationships.



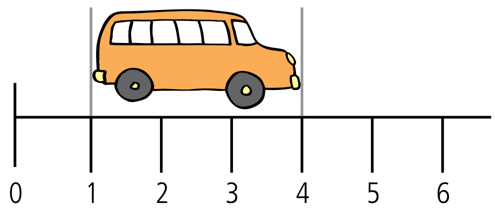
For example, in answering the question, "*Where does 5 belong on the number line?*" students must pay attention to both the end points to determine the amount of space each number occupies. Other number line activities include providing students with benchmarks along the number line. For example, where does the number 13 belong on the number line below?



Was the benchmark 10 helpful? How did it help with your reasoning?

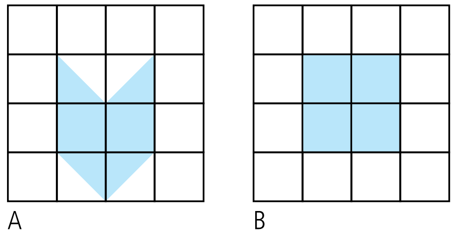
Tip: When introducing children to the number line, remember that many children are not accustomed to thinking about numbers as entities that occupy space (i.e., as intervals or units). As a result, many children see number line activities as practice in *counting on*, paying attention to ordinality but not the spatial relationships between numbers. Watch for students who determine their own intervals or units without acknowledging the end points.

Measurement: Measurement activities present a range of learning opportunities that require and engage spatial thinking.



How long is the school bus?

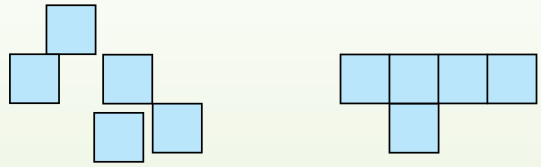
Elementary school students typically respond to this question in two ways: they either count the hash marks (arriving at an answer of four units) or they use the number aligned to the far right of the object as the marker of length (again, arriving at an answer of four units). As with standard measurement, answering this question requires attention to the intervals (spaces).



What shape takes up more blue space?

A common response to this question is that A occupies the larger area because it has six parts, while B has only four parts. As was the case with the linear measurement task above, this question requires an understanding of units as defined and consistent markers of space. Furthermore, this question requires students to reason spatially in one of two ways. One spatial solution involves embedding the larger triangle at the bottom of Figure A into place at the top of the figure. This visual-spatial approach allows for a direct comparison of the two figures (without any counting required). The other spatial approach to this problem involves the act of unitizing – forming a single and comparable unit by composing parts into a whole. In this case, square units can be made by combining two triangles.

Geometry and Spatial Sense: Geometry and spatial sense can be enhanced through puzzles and games. Beginning in the early years and continuing through all the grades, puzzles offer extensive opportunities to engage students' spatial thinking skills. Tangrams and pentominoes are two examples of materials that lend themselves to a variety of puzzle-like games and activities that develop skills, such as composing or decomposing shapes, transformational geometry (flips, transformations, rotations), visualization and congruence.



Using sets of five square tiles, how many unique pentomino configurations can you make? Note: A pentomino is a geometric figure formed by joining five equal squares edge to edge (see above example).

Patterning and Algebra: Early patterning experiences require both visual-spatial and numerical reasoning.

While the first problem above requires the identification of the core pattern (a visual process), the second problem requires the identification of both the core pattern and the pattern of rotation (a spatial process). A prompt for students could be, "What *could* come next?" This would open up a discussion about the various possibilities depending on the attribute (colour or orientation).

Graphing: Graphs of all kinds allow us to create visual displays of data. In the primary grades, this includes concrete graphs, pictographs and bar graphs. We can also graph algebraic expressions to show linear growing patterns. Graphs allow us to see the shape of our data and explore changes and rates of growth.

In the example below (from mathclips.ca), students use a tool to explore how changing values in the algebraic expression affects changes to the slope of the line and the *y*-intercept. Students can change the values in the algebraic or story representations to immediately see how the line changes. This powerful visual allows students to see connections between different representations and to understand how the input values change the appearance of the graph.

Algebraic Representation change to Story Rule Representation

Click the up/down buttons.

$y = 4x + 6$

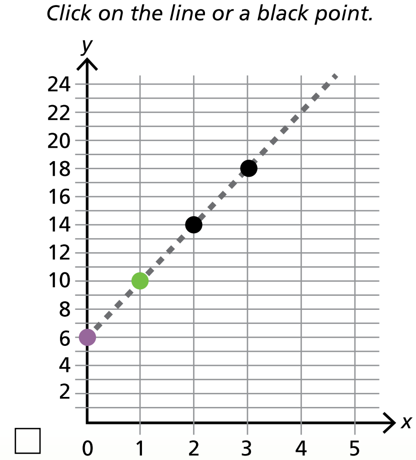
Story Representation change to Pictorial Representation

Click the up/down buttons.

Getting into a taxi costs $6 and it costs $4 more for every kilometre driven.

Graphical Representation

Drag the coloured points on the graph.
Click on the line or a black point.



Word Problems: Across the strands and grades, many problems deal with movement in space and require the problem solver to visualize key features of the problem.

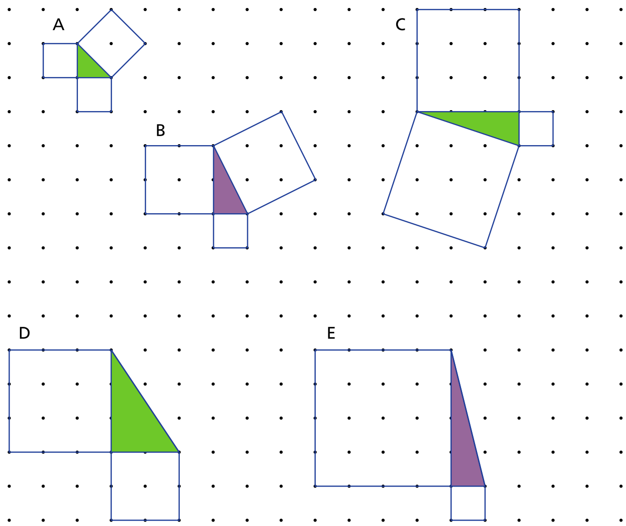
At each of the two ends of a straight path, a tree was planted, and then every 10 metres along the path another tree was planted. The length of the path is 30 metres. How many trees were planted altogether?

To solve this and other similar problems, it helps to either imagine or draw the spatial relationships described in the problem. Students can then connect their mathematical knowledge to the visual and contextual information to select appropriate strategies and to make reasonable estimates.

Intermediate and Senior

Students in intermediate and senior mathematics courses are required to apply their spatial reasoning in many contexts. These samples cross grades and courses.

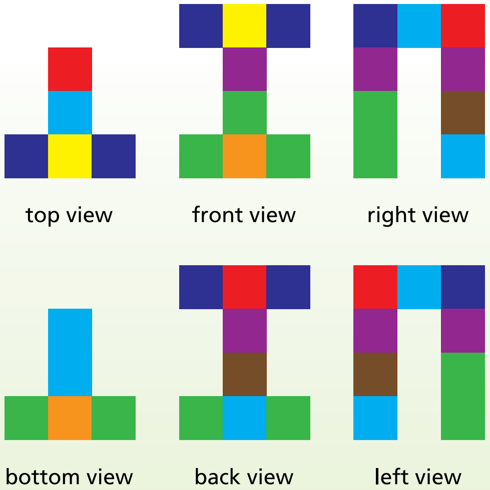
Seeing the Pythagorean Theorem: In this activity, students are asked to find and record the area derived from squaring the three sides of the right-angle triangles.



Using a visual-spatial approach will offer students an opportunity to "see" that the area of the square built on the hypotenuse (c^2) equals the sum of the areas of the squares ($a^2 + b^2$) built on the other two sides of the triangle. This visual-spatial representation helps students see meaning in the formula $a^2 + b^2 = c^2$ and to understand that the area of each square is determined from the side lengths of a, b, and c. This highlights the distinction between the area of the squares formed by the sides as seen in the relationship and the actual side lengths themselves.

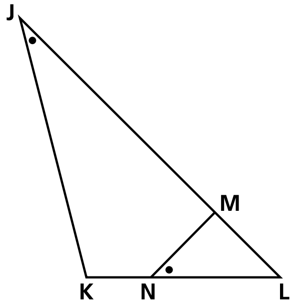
Using Perspective to Build a Three-Dimensional Structure: Students need to visualize how each of the perspectives below would combine to create a single three-dimensional figure. They need to view the figure from multiple sides and consider rotations of the image to match it with the figure they are envisioning.

Create the three-dimensional structure with the following views.

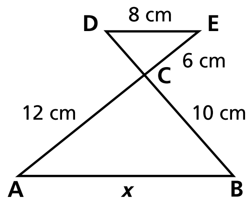


Visualizing Similar Triangles: The task on the left requires both a rotation and a reflection to disembed the two triangles, align the corresponding sides and angles, and be able to determine similarity. The task on the right requires students to determine equivalent angles and then rotate the triangles to align the corresponding sides and determine the unknown side length.

Identify two triangles that are similar. Justify how you know.



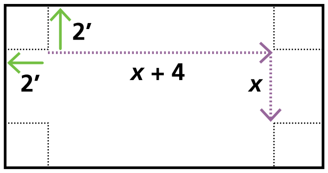
Find the length of x.



Creating a Quadratic Equation to Represent a Mathematical Context: Intermediate and senior students are required to create mathematics equations that represent complex mathematical situations.

John wants to build an open-top condenser pan for making maple syrup by cutting a square with a side length of two feet out of each corner of a rectangular piece of steel and folding up the sides. To make it stable, John wants the length of the pan to be four feet longer than the width. If the volume of the pan must be 64 cubic feet, how large should his piece of steel be?

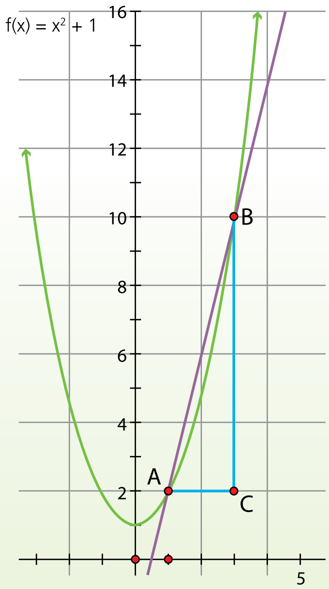
To relate the sheet of metal to the pan created, students must visualize how the sides would fold up and identify which measurements on the sheet correspond to the length, width and height of the newly created pan. To make the connection between the two-dimensional and three-dimensional aspects, students may create a diagram or model by using paper, which would look like the following:



Understanding Rates of Change: Students need to visualize the movement of point B along the graph of the function while simultaneously visualizing the corresponding change to the slope of the secant. They also need to establish the trend in the change of the slope of the secant as point B continues to move closer to point A, and they must connect this to the numeric value of the slope. This visualization will allow students to transition to the algebraic representation of this relationship.

Imagine point B moving along f(x)*, approaching point A. How is the slope of the secant line AB changing? What is the slope of AB when point B is exactly on top of point A?*

(TIPS4RM MCV4U GSP sketch, Unit #1 – Slope of Secant Line AB; http://www.edugains.ca/newsite/math/curriculum/curriculum12mcv4u.html)



Establishing Trigonometric Equations: Constructing diagrams is an essential strategy for setting up and solving trigonometric equations.

Samantha is standing at the top of a lighthouse on top of a cliff and sees a boat anchored offshore. She measures the angle of depression to be 38°. She descends the lighthouse and measures the angle of depression to the boat to be 25°. The lighthouse is 49 m tall. How far is the boat from the base of the cliff?

In this three-dimensional example, students need to consider both the location and the direction of the various items presented in the problem. They need to correctly place the measures and angles into the model or diagram. Additional calculations will be required before the diagram can be labelled, so a clear understanding of the location and alignment of the various points will ensure accuracy. For example, the angle of depression will need to be subtracted from 90° to obtain an angle measure for an interior angle. This step requires students to consider a horizontal line of sight and a downward rotation from the lighthouse perspectives. Furthermore, a reasonably proportional diagram allows for reasonable estimates.

How Can We Promote Spatial Reasoning?

We know spatial thinking is important, and we know that it can be improved through education and experience. We also know that spatial reasoning is not a separate content area or strand of mathematics, nor is it confined to geometry and spatial sense, but rather that it is a process that can support learning and communicating across the strands (and even in subjects beyond mathematics). In addition, we know that this is new ground; how do we get started in bringing this awareness into our practice?

1. Understand what spatial thinking is, and think of ways to support it within the content that you are already teaching.*

As a starting place, consider all the ways in which your current practice emphasizes the role of spatial thinking. Take the time to consider the ways in which various mathematics content is spatial in nature and find ways to *spatialize* this content where appropriate. Many mathematics activities naturally lend themselves to spatial strategies for solving problems. This idea doesn't require radical changes to mathematics programming; rather, what is required is "a shift in what information is being attended to and privileged" (Drefs & D'Amour, 2014).

For example, consider the following word problem:

> *What is the minimum number of toothpicks needed to build 100 adjoining toothpick squares that form a grid?*

One approach to solving this problem involves imagining or drawing a schematic representation of the problem. This visual-spatial approach to thinking about the problem provides a scaffold in which to carry out the calculations. Grounding this problem in a visual-spatial representation helps circumvent the often quick and incorrect solution of 400 toothpicks (1 square = 4 toothpicks, $100 \times 4 = 400$). Many word problems deal with movement in space and require the problem solver to visualize key features of the problem. To solve these problems, it helps to either imagine or draw the spatial relationships described in the problem.

* Several of the points in this section were adapted from Newcombe (2010, 2013).

It might also help to consider your own comfort level with spatial reasoning and how this translates in the classroom. A recent study suggests that teachers' own comfort level with spatial reasoning is related to their students' growth in spatial skills throughout the school year (Gunderson, Ramirez, Beilock & Levine, 2013). Students in Grade 1 and Grade 3 whose teachers had lower levels of anxiety about spatial reasoning were more likely to demonstrate significant gains in their spatial thinking skills by the end of the school year. The authors commented that because spatial thinking is not a stand-alone part of the curriculum (like reading or math), teachers high in anxiety about spatial reasoning might avoid incorporating spatial activities in the classroom. This study suggests that one way to improve students' spatial thinking skills might be to increase teachers' familiarity and comfort level with the teaching and learning of spatial thinking skills.

2. Emphasize the strand of geometry and spatial sense.

Research has shown that, in North America, the most inherently spatial area of mathematics – geometry and spatial sense – typically gets less time and attention in the classroom than other mathematics topics and strands (a recent large-scale survey showed this trend was also consistent in Ontario; see Bruce, Moss & Ross, 2012). Educators can boost opportunities to develop students' spatial thinking by providing greater focus in geometry in their mathematics programming. In fact, the National Council of Teachers of Mathematics (NCTM, 2006) has recommended that *at least half* of mathematics teaching and learning in the early years be focused on geometry, measurement and spatial reasoning (see Sinclair & Bruce, 2014). In particular, Sinclair and Bruce (2014) recommend that special attention be given to dynamic and transformational geometry (relating to objects in motion) rather than emphasizing static (non-moving) shapes and features of shapes. A focus on dynamic geometry fosters three areas of spatial thinking connected to mathematics performance: spatial visualization, perspective taking and mental rotation (Mix & Cheng, 2012).

3. Emphasize spatial language.

Teach and model the use of precise spatial words at every opportunity. Research has shown the importance of this in studies demonstrating that parents' use of spatial words correlated to their children's spatial ability and in studies where children who were taught spatial language performed better on spatial tasks than children who were not (see Tepylo, Moss & Hawes, 2014). For younger students, this language will include words related to location, distance, orientation and direction, for example, left, right, over, under, above, below, middle, parallel, tall and short. For older students, this language will involve the geometric vocabulary of rotations, translations and transformations.

Try This!

Master–Builder

Master–builder games provide playful opportunities to employ spatial thinking while also using the language of location and orientation. For this game, students work in pairs, with a barrier between them on the desk or table. On one side of the barrier, out of sight of the partner, one of the pair (the master) creates a visual design or pattern with objects, such as pattern blocks. The master then gives verbal instructions to his or her partner (the builder) to recreate the design. This game encourages rich and purposeful use of geometric language (including location, orientation, size, shape, distance, etc.). Many variations of this game are possible, depending on the materials students use.

4. Encourage visualization strategies.

Encourage students to use their visualization skills to better understand and solve problems. Provide plenty of opportunities for students to practise this important skill, and facilitate class-wide discussions about the process. Open discussions that allow students to share how they visualize problems and solutions emphasize the importance of the imagination in mathematics and that there are many different ways of imagining a problem and its solutions. It is important for students to recognize that not all perspectives are the same, and some ways of seeing a problem are more effective than others. While a common practice in our mathematics classrooms involves having students predict or estimate the solution to the problem, often what we are really asking students to do is visualize. By making students aware of visualization and providing opportunities to practise and develop the skill, we give students yet another strategy to rely on when problem solving.

Not all students will naturally rely on spatial strategies to solve mathematical problems. For some students, mathematics is perceived as an activity based purely in numbers. Even in geometry – the science of spatial relationships – students often feel pressured to produce an answer through formulaic procedures and numeration alone. Students need to be encouraged to reason about mathematics by using a variety of approaches, including spatial strategies. Students also need to be encouraged to use their imaginations and visualize mathematics problems and solutions. Just as visualization helps with reading and writing, the same is true of mathematics. Students could be asked, for example, to close their eyes and see a translation to a figure on a Cartesian plane to predict its new location and orientation. They could then test their prediction to see if they were correct.

Try This!

Visualization Questions and Prompts

To have students articulate the visual imagery they are using when problem solving, try asking students, "What are you seeing in your mind?" or "What did you see or visualize that helped you to solve the problem?" These kinds of questions elicit descriptions from students that clearly show their understanding while encouraging them to think metacognitively about their visualization strategies.

5. Emphasize and celebrate visual displays of data.

Our classrooms already contain an abundance of visual data, so we know that this is a good way to encourage spatial thinking. Diagrams, maps and graphs of all kinds are important and powerful representations that can be featured prominently in the classroom. Nora Newcombe (2013) suggests that we can be intentional in the ways that we display many kinds of information, such as by showing daily schedules in which smaller blocks of time take up less space, "reinforcing the idea that graphic variation in spacing can have real meaning" (p. 29). She also suggests using visuals to help students see and compare very large and very small objects (such as atoms) and quantities (such as time).

Encouraging students to represent their mathematical thinking is also valuable, of course (we have long been asking students to represent their thinking in multiple ways, using pictures, numbers and words). But Doug Clements and Julie Sarama (2009) point out that not all visuals are helpful. For example, some research has shown that drawings created by high-achieving students in solving mathematics problems tend to show spatial relationships involved in the problem more accurately (and are therefore more helpful in solving the problem); the images created by lower-achieving students, on the other hand, may not actually help these students solve the problem (these drawings tend to focus on surface features of the problem rather than representing general or abstract ideas). Rather than simply being told to include drawings, many students would benefit from explicit support in the creation of visuals, as well as instructional support to develop awareness of and experience with the kinds of visuals and schematics that scaffold logic in problem solving (Venn diagrams, for example).

6. Use gestures and encourage students to use gestures.

Spatial thinking, like other cognitive processes, can at times appear invisible to the mathematics educator. The fact that spatial thinking can occur in the absence of language can cause difficulties in the communication of ideas and solutions. For this reason, students – especially young students – are not always able to offer verbal explanations for spatial solutions to problems. For example, a child might be able to compose a hexagon in a variety of shapes and ways yet experience difficulties articulating the process. Students need opportunities to explain their reasoning while educators encourage and watch for other expressions of understanding. For example, gesturing – that is, communicating ideas through the use of the hands – is an often-used yet often-overlooked form of communication. Gesture is an especially powerful means of expressing spatial information and provides information to the listener that is not always expressed in words. When people are explaining a spatial concept (giving directions, for example, or explaining geometric terms, such as rotation, translation and transformation), they are likely to augment speech with the use of gesture. In fact, several studies in mathematics education show that the use of gestures helps students build understanding; both teacher and student use of gestures have been shown to correlate strongly with student achievement (Goldin-Meadow, 2005). This research, as well as other research demonstrating the links between finger use and brain activity (Dehaene, Piazza, Pinel & Cohen, 2003), shows that gestures may be incredibly powerful in helping form pathways in the brain and in the development of conceptual understandings, and requires further attention.

7. Provide meaningful opportunities to investigate mathematical concepts and problems by using manipulatives.

We have long known the importance of manipulatives for students in building their conceptual understanding in mathematics. The new research on spatial reasoning adds another layer of importance to their use, even as it helps us to understand how they support student learning. The use of tools, for example, is a highly spatial activity. Newcombe (2013) points out that the development and use of tools (a key evolutionary moment for humans and one of the "hallmarks of our species") relies on spatial thinking: "to create a successful tool, one must first imagine a shape that is relevant to a particular function, such as cutting or digging, and then fashion that shape out of larger forms" (p. 102). We can see how the use of manipulatives in the mathematics classroom can help to consolidate understanding and concept development as visualization and problem solving are inherent in their use. Further, the research on the importance of gestures provides some insight as to how kinesthetic interactions with materials may form pathways in our brains to help us understand and communicate.

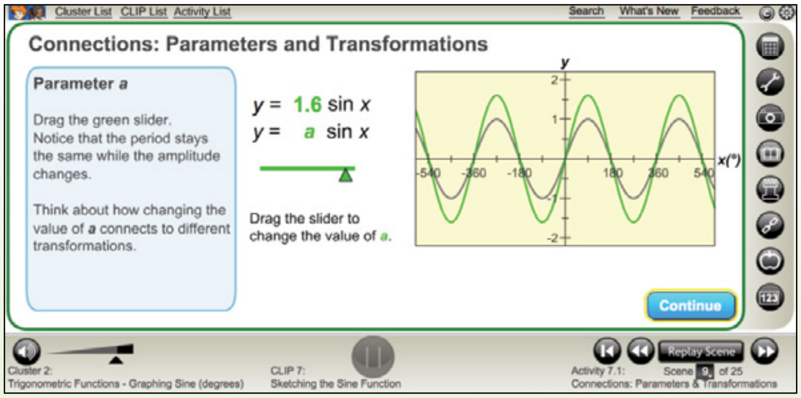
Although it is important that manipulatives are made available to students, an even more critical consideration is how to ensure the use of manipulatives in meaningful ways – as integral to the thinking and the problem solving. In other words, the learning task is designed so that manipulatives are not just used to communicate or show representations of thinking after the cognitive work of the problem solving is done; they are the tools with which the problem is solved. Consider the following example for exploring linear functions: students are asked to build growing patterns by using square tiles, make algebraic rules to generalize about the growth of their patterns, then make concrete graphs by using the actual tiles and finally create a standard line graph to show the growth of their patterns (see Beatty & Beatty, 2012). In this example, the tiles are the site of problem solving and representation; they are integral to the task, not an add-on or an option for students if they choose. This is an example of a powerful use of manipulatives that builds understanding through visual and kinesthetic means by providing hands-on exploration of numeric quantities and algebraic expressions. (See Ontario Ministry of Education, 2013.) This use of manipulatives is very similar to what mathematicians do when they work with models or simulations to solve problems. It also demonstrates that the power of manipulatives is in helping us to move between concrete representations and abstract ideas, helping students to visually understand and internalize abstract concepts.

8. Provide playful opportunities for students to exercise their spatial reasoning.

Many playful activities require spatial thinking; think of jigsaw puzzles, many board games, guided play with blocks or other geometric shapes, and some types of video games. A solid body of research has established connections between these kinds of playful activities and spatial reasoning, and in some cases, also mathematics performance (see Tepylo, Moss & Hawes, 2014). Puzzle play and block building, especially in a semi-structured or guided play context, have been shown to improve spatial performance and geometric knowledge (Casey, Andrews et al., 2008; Casey, Erkut et al., 2008; Fisher, Hirsh-Pasek, Newcombe & Golinkoff, 2013) and mathematics performance (Clements & Sarama, 2009). Tetris and first-person role-playing games in which the player moves through virtual environments have been shown to be strongly linked to improvements in spatial reasoning (Feng et al., 2007; Terlecki, Newcombe & Little, 2008). New research on apps that capitalize on the use of touchscreen technology to foster gestures along with spatial reasoning show a lot of promise (Sinclair & Bruce, 2014). Much of this research affirms that time spent in this kind of play is time well spent when it comes to spatial reasoning.

9. Take advantage of technology.

Digital technologies allow us to manipulate and see space and spatial relationships like never before. GIS, GPS, Google Earth, and other computer models, tools and interfaces (such as interactive whiteboards) allow us to manipulate objects and ideas in ways we could never have done with pencil and paper or chalk and chalkboard. Touchscreen technologies foster gestures and spatial reasoning to build conceptual understanding (Bruce, 2014b). In the example below, a student can drag the slider to change values, showing how a change in parameters affects the graph in a periodic function. From JK to Grade 12, technology presents opportunities for students to see and even manipulate mathematical ideas in powerful ways. Understanding the importance of spatial reasoning in mathematics places even greater imperative on us as educators to take advantage of technology when and where it fits.



Visit www.mathclips.ca for other examples.

Ministry Resources and References

Ministry Resources

Ontario Ministry of Education. (2005a). *The Ontario curriculum: Grades 1–8 mathematics*. Toronto, ON: Queen's Printer for Ontario. http://www.edu.gov.on.ca/eng/curriculum/elementary/math18curr.pdf

Ontario Ministry of Education. (2005b). *The Ontario curriculum: Grades 9 and 10 mathematics*. Toronto, ON: Queen's Printer for Ontario. http://www.edu.gov.on.ca/eng/curriculum/secondary/math910curr.pdf

Ontario Ministry of Education. (2007). *The Ontario curriculum: Grade 11 and 12*. Toronto, ON: Queen's Printer for Ontario. http://www.edu.gov.on.ca/eng/curriculum/secondary/math1112currb.pdf

Ontario Ministry of Education. (2010). *Full-day early learning Kindergarten program* (draft version). Toronto, ON: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/kindergarten_english_june3.pdf

EduGAINS. (n.d.). *Student critical learning instructional paths supports (CLIPS) in mathematics: Grades K–12*. Retrieved from http://www.edugains.ca

EduGAINS. (n.d.) TIPS4RM. http://www.edugains.ca

Ontario Ministry of Education. (2012). *Paying attention to proportional reasoning*. Toronto, ON: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/teachers/studentsuccess/ProportionReason.pdf

Ontario Ministry of Education. (2013). *Paying attention to algebraic reasoning*. Toronto, ON: Queen's Printer for Ontario. Retrieved from http://www.edu.gov.on.ca/eng/literacynumeracy/PayingAttentiontoAlgebra.pdf

References and Related Readings

Ansari, D. (2013). *Disorders of the "mathematical brain": Developmental dyscalculia and mathematics anxiety.* Keynote address: Ontario Association of Mathematics Educators, Seneca College, Toronto, ON.

Beatty, R., & Bruce, C. (2012). *Linear relationships: From patterns to algebra*. Toronto, ON: Nelson.

Bruce, C. (2014a). "Spatializing" the mathematics curriculum: New perspectives, new frameworks. *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education*. Vancouver, BC: PME.

Bruce, C. (2014b). Use of the iPad as a mediator for the development of spatial reasoning in young children. *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education*. Vancouver, BC: PME.

Bruce, C., Moss, J., & Ross, J. (2012). *Survey of JK to Grade 2 teachers in Ontario, Canada: Report to the Literacy and Numeracy Secretariat of the Ministry of Education*. Toronto, ON: Ontario Ministry of Education.

Casey, B., Andrews, N., Schindler, H., Kersh, J. E., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. *Cognition and Instruction, 26*(3), 269–309.

Casey, B., Erkut, S., Ceder, I., & Young, J. (2008). Use of a storytelling context to improve girls' and boys' geometry skills in kindergarten. *Journal of Applied Developmental Psychology, 29*, 29–48.

Center for Spatial Studies, University of Southern California, Santa Barbara. (n.d.). About us: Imagine a nation of spatial thinkers. Retrieved from http://spatial.ucsb.edu/about-us/

Cheng, Y. L., & Mix, K. S. (2012). Spatial training improves children's mathematics ability. *Journal of Cognition and Development*. Advance online publication. doi:10.1080/15248372.2012.725186

Clements, D., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York, NY: Routledge.

Clements, D. H., & Sarama, J. (2011). Early childhood teacher education: The case of geometry. *Journal of Mathematics Teacher Education, 14*, 133–148.

Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2003). Three parietal circuits for number processing. *Cognitive Neuropsychology, 20*, 487–506.

Drefs, M., & D'Amour, L. (2014). The application of ambiguous figures to mathematics: In search of the spatial components of number. *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education*. Vancouver, BC: PME.

Farmer, G., Verdine, B. N., Lucca, K., Davies, T., Dempsey, R., Hirsh-Pasek, K., & Golinkoff, R. M. (2013). *Putting the pieces together: Spatial skills at age 3 predict to spatial and math performance at age 5*. Presented at the Society for Research in Child Development, Seattle, WA.

Feng, J., Spence, I., & Pratt, J. (2007). Playing an action video game reduces gender differences in spatial cognition. *Psychological science, 18*(10), 850–855.

Fisher, K. R., Hirsh-Pasek, K., Newcombe, N., & Golinkoff, R. M. (2013). Taking shape: Supporting preschoolers' acquisition of geometric knowledge through guided play. *Child development, 84*(6), 1872–1878.

Goldin-Meadow, S. (2005). *Hearing gesture: How our hands help us think*. Cambridge, MA: Harvard University Press.

Gunderson, E. A., Ramirez, G., Beilock, S. L., & Levine, S. C. (2013), Teachers' spatial anxiety relates to 1st- and 2nd-graders' spatial learning. *Mind, Brain, and Education, 7*, 196–199. doi:10.1111/mbe.12027

Huttenlocher, J., Levine, S., & Vevea, J. (1998). Environmental input and cognitive growth: A study using time-period comparisons. *Child Development, 69*(4), 1012–1029.

Johnson, C. Y. (2014, January 28). A not-close-enough view of "Hawking." *The Boston Globe*. Retrieved from https://www.bostonglobe.com/arts/television/2014/01/28/hawking-doc-brings-almost-close-enough-his-genius/UAbkAEKXaYKxVAY0jAkT8J/story.html

Jordan, N., & Levine, S. (2009). Socioeconomic variation, number competence, and mathematics learning difficulties in young children. *Developmental Disabilities Research Reviews, 15*, 60–68.

Larsen, K. (2005). *Stephen Hawking: A biography*. Westport, CT: Greenwood Press.

Lohman, D. F. (1996). Spatial ability and G. In I. Dennis & P. Tapsfield (Eds.), *Human abilities: Their nature and assessment* (pp. 97–116). Hillsdale, NJ: Erlbaum.

Mix, K. S., & Cheng, Y.-L. (2012). The relation between space and math: Developmental and educational implications. In J. B. Benson (Ed.), *Advances in child development and behavior* (Vol. 42, pp. 197–243). San Diego, CA: Academic Press.

National Research Council. (2006). *Learning to think spatially: GIS as a support system in the K–12 curriculum*. Washington, DC: National Academic Press.

National Council of Teachers of Mathematics. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: Author.

Newcombe, N. S. (2010). Picture this: Increasing math and science learning by improving spatial thinking. *American Educator, 34*, 29–35.

Newcombe, N. S. (2013). Seeing relationships: Using spatial thinking to teach science, mathematics, and social studies. *American Educator, 37*(1), 26–31.

Newcombe, N. S., & Frick, A. (2010). Early education for spatial intelligence: Why, what, and how. *Mind, Brain, and Education, 4*(3), 102–111.

Reed, S. K. (2010). *Thinking visually*. New York, NY: Psychology Press.

Rourke, B. P. (1993). Arithmetic disabilities, specific and otherwise: A neuropsychological perspective. *Journal of Learning Disabilities, 26*, 214–226.

Sarama, J., & Clements, D. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York, NY: Taylor & Francis.

Shumway, J. F. (2013). Building bridges to spatial reasoning. *Teaching Children Mathematics, 20*(1), 44–51.

Sinclair , N., & Bruce, C. (2014). Spatial reasoning for young learners. *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education*. Vancouver, BC: PME.

Tepylo, D. H., Moss, J., & Hawes, Z. (forthcoming). Spatial reasoning: Considerations for mathematics educators.

Terlecki, M. S., Newcombe, N. S., & Little, M. (2008). Durable and generalized effects of spatial experience on mental rotation: Gender differences in growth patterns. *Applied Cognitive Psychology, 22*, 996–1013.

Trent University, School of Education and Professional Learning. Trent Math Education Research Collaborative. Retrieved from http://tmerc.ca

University of Cambridge, NRICH Enriching Mathematics, Millennium Mathematics Project. Retrieved from http://nrich.maths.org/frontpage

University of Toronto, Ontario Institute for Studies in Education (OISE). Robertson program for inquiry-based teaching in math and science. Retrieved from http://www.oise.utoronto.ca/ robertson/index.html

Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., & Newcombe, N. S. (2013). The malleability of spatial skills: A meta-analysis of training studies. *Psychological Bulletin, 139*(2), 352.

Verdine, B. N., Golinkoff, R. M., Hirsh-Pasek, K., Newcombe, N. S., Filipowicz, A. T., & Chang, A. (2013). Deconstructing building blocks: Preschoolers' spatial assembly performance relates to early mathematical skills. *Child Development*. Advance online publication. doi:10.1111/cdev.12165

Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology, 101*, 817–835.

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